TRANSIENT RESPONSE OF THE 1-ST ORDER DYNAMIC SYSTEM WITH FLUCTUATING PARAMETERS

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The paper deals with the analysis of the dynamic behaviour of the 1-st order system with two random parameters. The theoretical results have been compared with experiments on flow model of a stirred tank reactor.

In several recent papers[1-3] the dynamic behaviour of the system with random parameter has been analyzed. It has been found that it may differ under certain conditions significantly from the behaviour of the corresponding system with mean value of random parameter.

Herles[4] analyzed recently a 1-st order system with random time constant and derived theoretically the basic statistical characteristics of the impulse response. In presented paper statistical characteristics are derived for the step response of the 1-st order system with two random parameters for special types of probability distribution. In view of the difficulties in realization of the pure impulse response the results obtained are more convenient for the experimental verification. The results of the experiments on a model physical system are presented and the comparison with theoretical analysis is carried out.

THEORETICAL

Statistical characteristics of the step response

Let us have a 1^{st} order dynamic system described by Eq. (*1*):

$$(y'/a) + y = bx\,, \tag{1}$$

where x, y are input and output time function of the system. The coefficients a, b randomly vary in time under following conditions:

$$\left.\begin{array}{l} a) \text{ The random changes of parameter } a \text{ can occur only at discrete time instants } t_i \; (i = 0, 1, 2, \ldots; t_0 = 0) \\ b) \text{ Within the interval } t_i \leqq t < t_{i+1} \text{ the parameter } a \text{ keeps the last value } a_i \text{ unchanged} \\ c) \text{ The parameter } b \text{ is the random variable, but having a constant value in one realization} \\ d) \text{ The system response } y(t) \text{ is a continuous function of time.} \end{array}\right\} \quad (2)$$

Such system can be solved as time invariant within every interval $\langle t_i; t_{i+1})$:

$$y' + a_i y = a_i b x \,. \tag{3}$$

The solution is then given:

$$y(t) = \exp\left(-\int a_i \, \mathrm{d}t\right) \cdot \left[a_i b \int x(t) \exp\left(\int a_i \, \mathrm{d}t\right) \mathrm{d}t + k_i\right] =$$

$$= \exp\left(-a_i t\right) \cdot \left[a_i b \int x(t) \exp\left(a_i t\right) \mathrm{d}t + k_i\right], \tag{4}$$

$$t_i \leqq t < t_{i+1}$$

where k_i is the integration constant, having a different value in every interval $\langle t_i, t_{i+1}\rangle$. For the unit step input

$$\begin{aligned} x(t) &= 1 \qquad t \geqq 0 \\ &= 0 \qquad t < 0 \end{aligned} \tag{5}$$

the solution equals ($t \geqq 0$):

$$y(t) = \exp\left(-a_i t\right) \cdot \left[a_i b \int \exp\left(a_i t\right) \mathrm{d}t + k_i\right] = b + k_i \exp\left(-a_i t\right), \tag{6}$$

$$0 \leqq t_i \leqq t < t_{i+1} \,.$$

The unknown values k_i can be found using the continuity assumption (*2d*) in the time instants t_i (y_i denotes the values of the output in those instants):

$$\begin{aligned} y_{0-} &= 0 \\ &= y_{0+} = b + k_0 \Rightarrow k_0 = -b \end{aligned} \tag{7}$$

$$\begin{aligned} y_{1-} &= b + k_0 \exp(-a_0 \Delta t) = \\ &= y_{1+} = b + k_1 \exp(-a_1 \Delta t) \\ &\vdots \\ y_{i-} &= b + k_{i-1} \exp(-a_{i-1}\, i\, \Delta t) = \\ &= y_{i+} = b + k_i \exp(-a_i\, i\, \Delta t) \end{aligned} \tag{8}$$

$$\begin{aligned} k_i &= \frac{\exp(-a_{i-1}\, i\, \Delta t)}{\exp(-a_i\, i\, \Delta t)} k_{i-1} = \\ &= \frac{\exp(-a_{i-1}\, i\, \Delta t)}{\exp(-a_i\, i\, \Delta t)} \cdot \frac{\exp(-a_{i-2}(i-1)\,\Delta t)}{\exp(-a_{i-1}(i-1)\,\Delta t)} k_{i-2} = \\ &= \ldots\ldots \\ &= \exp(a_i\, i\, \Delta t) \exp(-\Delta t \sum_{j=0}^{i-1} a_j)\, k_0 \,. \end{aligned} \tag{9}$$

Substituting in Eq. (6):

$$\begin{aligned} y_i &= b[1 - \exp(a_i\, i\, \Delta t) \exp(-\Delta t \sum_{j=0}^{i-1} a_j) \exp(-a_i\, i\, \Delta t)] = \\ &= b[1 - \exp(-\Delta t \sum_{j=0}^{i-1} a_j)] \,. \end{aligned} \tag{10}$$

Supposing that the random variables b, a_j ($j = 0, 1 \ldots$) are mutually independent, the mean and the variance of the response are given ($p(.)$ denotes the probability density):

$$\begin{aligned} E\{y_i\} &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} b[1 - \exp(-\Delta t \sum_{j=0}^{i-1} a_j)] \,. \\ &\quad . \, p(b, a_0, \ldots, a_{i-1})\, db\, da_0 \ldots da_{i-1} \\ &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} b[1 - \exp(-\Delta t \sum_{j=0}^{i-1} a_j)] \,. \\ &\quad . \, p(b)\, p(a_0) \ldots p(a_{i-1})\, db\, da_0 \ldots da_{i-1} \\ &= E\{b\} \left[1 - \prod_{j=0}^{i-1} \int_{-\infty}^{\infty} \exp(-a_j \Delta t)\, p(a_j)\, da_j \right] . \end{aligned} \tag{11}$$

Supposing the same distribution for all a_j (stationarity), we obtain

$$E\{y_i\} = \bar{y}_i = \bar{b} \left\{1 - \left[\int_{-\infty}^{\infty} \exp(-a_j \Delta t)\, p(a_j)\, da_j\right]^i\right\} \tag{12}$$

Similarly the variance:

$$D\{y_i\} = E\{y_i^2\} - \bar{y}_i^2 =$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} b^2[1 - 2 \exp(-\Delta t \sum_{j=0}^{i-1} a_j) + \exp(-2 \Delta t \sum_{j=0}^{i-1} a_j)] \,.$$

$$. \, p(b)\, p(a_0) \dots p(a_{i-1})\, db\, da_0 \dots da_{i-1}$$

$$-\bar{b}^2 \left\{1 - 2\left[\int_{-\infty}^{\infty} \exp(-a_j \Delta t) \,.\, p(a_j)\, da_j\right]^i + \left[\int_{-\infty}^{\infty} \exp(-a_j \Delta t) \,.\, p(a_j)\, da_j\right]^{2i}\right\}$$

$$= \underbrace{(E\{b^2\} - \bar{b}^2)}_{D\{b\}} \left\{1 - 2\left[\int_{-\infty}^{\infty} \exp(-a_j \Delta t) \,.\, p(a_j)\, da._j\right]^i + \left[\int_{-\infty}^{\infty} \exp(-2a_j \Delta t) \,.\, p(a_j)\, da_j\right]^i\right\}$$

$$+ \bar{b}^2 \left\{\left[\int_{-\infty}^{\infty} \exp(-2a_j \Delta t) \,.\, p(a_j)\, da_j\right]^i - \left[\int_{-\infty}^{\infty} \exp(-a_j \Delta t) \,.\, p(a_j)\, da_j\right]^{2i}\right\}. \quad (13)$$

Similar integrals as in Eq (*12*), (*13*) have been solved by Herles[4] for the case of normal and uniform distribution of a_j

a) normal distribution

$$p(a_j) = \{1/[\sigma_a \sqrt{(2\pi)}]\} \exp[-(a_j - \bar{a})^2/(2\sigma_a)] \quad (14)$$

$$I = \int_{-\infty}^{\infty} \exp(-a_j \Delta t) \,.\, p(a_j)\, da_j = \exp\left[-\left(\bar{a} - \frac{\sigma_a^2 \Delta t}{2}\right)\Delta t\right] \quad (15)$$

b) uniform distribution:

$$p(a_j) = 1/(2b) \quad \text{for} \quad \bar{a} - b < a_j \leqq \bar{a} + b \quad (16)$$

$$= 0 \quad \text{for other } a_j$$

$$I = \exp(-\bar{a}\, \Delta t) \,.\, \left(\frac{\sinh b\, \Delta t}{b\, \Delta t}\right) \quad (17)$$

$$\sigma_a = b/\sqrt{3} \tag{18}$$

c) binary distribution:

Random variables a_j can acquire only two values ($a_j = \bar{a} \pm \sigma_a$), each of them with probability $P = 1/2$:

$$p(a_j) = \tfrac{1}{2}[\delta(\bar{a} - \sigma_a) + \delta(\bar{a} + \sigma_a)] \tag{19}$$

($\delta(.)$ is Dirac delta)

$$I = \exp(-\bar{a}\,\Delta t) \,.\, \cos h\, \sigma_a\, \Delta t \tag{20}$$

Using expressions for I in (*12*), (*13*), we obtain:

a) normal distribution:

$$\bar{y}_i/\bar{b} = 1 - \exp\left[-\left(\bar{a} - \frac{\sigma_a{}^2\,\Delta t}{2}\right) i\,\Delta t\right] \tag{21}$$

$$(\sigma_{yi}/\bar{b})^2 = \exp(-2\bar{a}\, i\,\Delta t)\exp(2\sigma_a^2\, i\,\Delta t^2) - \exp(\sigma_a^2\, i\,\Delta t^2) + \\ + (\sigma_b/\bar{b})^2 \left\{1 - 2\exp\left[-\left(\bar{a} - \frac{\sigma a^2\,\Delta t}{2}\right) i\,\Delta t\right] + \exp\left[-(\bar{a} - \sigma_a^2\,\Delta t)\, 2i\,\Delta t\right]\right\} \tag{22}$$

b) uniform distribution:

$$\bar{y}_i/\bar{b} = 1 - \exp(-\bar{a}\, i\,\Delta t) \,.\, \left(\frac{\sinh \sigma_a\,\Delta t\,\sqrt{3}}{\sigma_a\,\Delta t\,\sqrt{3}}\right)^i \tag{23}$$

$$(\sigma_{yi}/\bar{b})^2 = \exp(-2\bar{a}\, i\,\Delta t)\left[\left(\frac{\sinh \sigma_a\,\Delta t\, 2\sqrt{3}}{\sigma_a\,\Delta t\, 2\sqrt{3}}\right)^1 - \\ - \left(\frac{\sinh \sigma_a\,\Delta t\,\sqrt{3}}{\sigma_a\,\Delta t\,\sqrt{3}}\right)^{2i}\right] + \\ + (\sigma_b/\bar{b})^2\left[1 - 2\exp(-\bar{a}\, i\,\Delta t)\left(\frac{\sinh \sigma_a\,\Delta t\,\sqrt{3}}{\sigma_a\,\Delta t\,\sqrt{3}}\right)^i + \\ + \exp(-2\bar{a}\, i\,\Delta t)\left(\frac{\sinh \sigma_a\,\Delta t\, 2\sqrt{3}}{\sigma_a\,\Delta t\, 2\sqrt{3}}\right)^i\right]. \tag{24}$$

c) binary distribution

$$\bar{y}_i/\bar{b} = 1 - \exp(-\bar{a}\,i\,\Delta t)\,.\,\cosh^i \sigma_a\,\Delta t \quad (25)$$

$$(\sigma_{yi}/\bar{b})^2 = \exp(-2\bar{a}\,i\,\Delta t)\left[\cosh^i 2\sigma_a\,\Delta t - \cosh^{2i} \sigma_a\,\Delta t\right] + \\ + (\sigma_b/\bar{b})^2\left[1 - 2\exp(-\bar{a}\,i\,\Delta t)\,.\,\cosh^i \sigma_a\,\Delta t + \right. \\ \left. + \exp(-2\bar{a}\,i\,\Delta t)\,.\,\cosh^i 2\sigma_a\,\Delta t\right]. \quad (26)$$

The normalized standard deviation and the difference between the stochastic and deterministic step responses for all three distribution are shown on Fig. 1 ($\sigma_a/\bar{a} = 0{\cdot}5$; $\sigma_b/\bar{b} = 0{\cdot}1$; $\bar{a}\,\Delta t = 0{\cdot}2; 1$). It illustrates (similarly to Herles[4]), that even a very large amount of fluctuations exhibits relatively very small influence on the mean value of the response. The difference is much less than the standard deviation of the response. The influence of the type of distribution is almost negligible (some smaller differences for $\bar{a}\,\Delta t = 1$ have less practical importance due to very rough approximation of a real "continuous" case of fluctuating parameters).

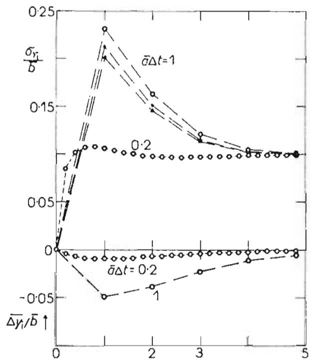

FIG. 1

Theoretical Normalized Standard Deviation of the Step Response (up) and the Difference (down) Between the Responses of the Deterministic and Stochastic Systems (mean value)

○ normal distribution; + uniform distribution; △ binomial distribution; (coinciding values at $\bar{a}$ $\Delta t = 0{\cdot}2$ are indicated as "normal").

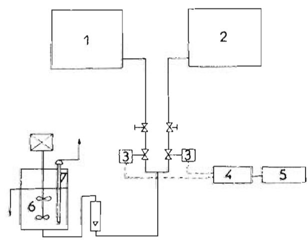

FIG. 2

Diagram of the Experimental Unit

1, *2* NaCl solution storage tanks, *3* solenoid valves, *4* power transducer, *5* random process generator GENAP III, *6* stirred reactor, *7* conductivity probe.

EXPERIMENTAL

Experiments with flow model of a continuous stirred tank reactor have been carried out and concentration step input response obtained in the experimental arrangement acc. to Fig. 2. Tank reactor of the constant temperature of volume $V = 14{\cdot}2$ l was filled initially with distilled water. Perfect mixing of the content of reactor was attained by mechanical stirrer. The mean flow-rate of the solution of NaCl 1·5 kg/m^3 concentration in distilled water was determined as $\bar{Q} = 170{\cdot}4$ l/h.

This corresponds to the mean value of time constant $\bar{a} = 0{\cdot}2\ \text{min}^{-1}$. For different values of relative standard deviation $\sigma_a/\bar{a}$ corresponding values of two different flow rates Q_1, Q_2 of the solution were calculated. At $\sigma_a/\bar{a} = 0{\cdot}5 Q_1 = 255{\cdot}6$ l/h, $Q_2 = 85{\cdot}2$ l/h. Generator of random processes GENAP III was used for generation of true random two-level sequence with binomial distribution and identical probability of "0" and "1".

This sequence controlled through power transducer and solenoid valves switching from one value of the flow rate to the other.

Two different switching intervals were chosen, namely $\Delta t = 1$ min and $\Delta t = 5$ min. For this arrangement the governing equation can be written as

$$(V/Q)\, dc_2/dt + c_2(t) = c_1(t)$$

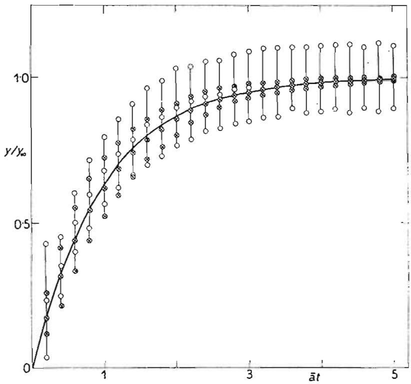

FIG. 3

Comparison of the Experimental Results with the Step Response of the Simplified Theoretical Model

○ Sample average and standard deviations from the experiments at $\bar{a}\,\Delta t = 0{\cdot}2$, $\sigma_a/\bar{a} = 0{\cdot}5$;

⊗ theoretical mean values and standard deviations resulting from the simplified model for $\bar{a}\,\Delta t = 0{\cdot}2$, $\sigma_a/\bar{a} = 0{\cdot}5$, $\sigma_b/\bar{b} = 0{\cdot}0$;

— deterministic step response.

or

$$(V/Q)\,\mathrm{d}E_2/\mathrm{d}t + E_2(t) = (K_1/K_2)\,E_1(t)\,. \tag{27}$$

Twenty experiments were performed using $\Delta t = 1$ min, twenty experiments using $\Delta t = 5$ min, at $\sigma_a/\bar{a} = 0{\cdot}5$. The estimates of average values and standard deviations were calculated at discrete intervals Δt.

Digital simulation for the same parameters as in the experiments has proceeded on 9821 A HP calculator using software generation of random binary sequence. Thus two sets of simulated runs were obtained and processed on the calculator in the same way as the experimental results.

The additional experiments were carried out which included the measurements of transient response at constant flow rate $Q = 170{\cdot}4$ l/h, 20 experiments at each of the two intervals $\Delta t = 1$ min, $\Delta t = 5$ min. In this way the separate effect of the random constant b could be estimated. The results were processed in the same way as before.

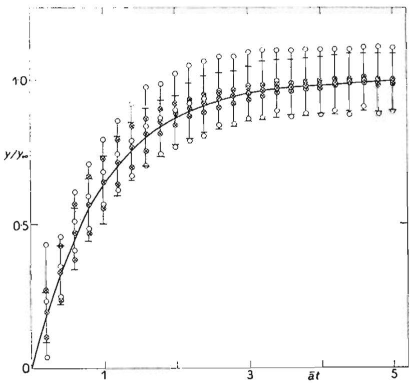

FIG. 4

Comparison of Experimental, Simulated and Theoretical Data $\sigma_a/\bar{a} = 0{\cdot}5$, $\bar{a}\,\Delta t = 0{\cdot}2$

○-○-○ Sample average and standard deviation values from experiments;

⊗-⊗-⊗ sample average and standard deviation values from simulation runs, $\sigma_b/\bar{b} = 0{\cdot}0$;

I theoretical standard deviation limits for $\sigma_b/\bar{b} = 0{\cdot}1$;

— deterministic step response.

RESULTS

In Fig. 3 the comparison of the theoretical results for parameters $\sigma_a/\bar{a} = 0{\cdot}5$, $\bar{a}\Delta t = 0{\cdot}2$ and $\sigma_b/\bar{b} = 0{\cdot}0$ indicates the inadequacy of this model when compared with experimental data. The normalized standard deviation values of the transient response dont tend to zero with increasing time as it would result from the theory, but to some constant value distinct from zero. Such behaviour of real experimental data can be better matched by the concept of fluctuating constant b.

In Figs 4 and 5 the experimental results together with the results from simulation runs are compared with the results of theoretical analysis for selected values of parameters within the range which is interesting from the practical point of view.

The deviation of mean values of transient responses at different time intervals from the deterministic solution as it results from the theoretical analysis is significant only for $\bar{a}\,\Delta t = 1$ and it agrees with both the experimental and simulation results.

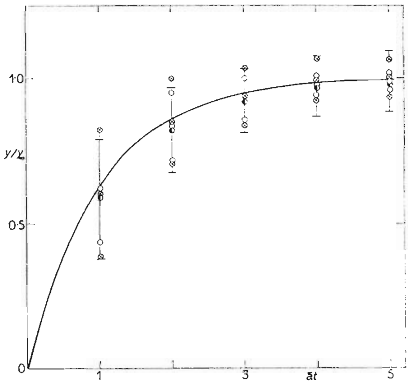

FIG. 5

Comparison of Experimental, Simulated and Theoretical Data
$\sigma_a/\bar{a} = 0{\cdot}5$, $\bar{a}\,\Delta t = 1{\cdot}0$;

⊗-⊗-⊗ sample average and standard deviation values from experiments;

O-O-O sample average and standard deviation values from simulation runs, $\sigma_b/\bar{b} = 0{\cdot}0$;

⊤-◐-⊥ theoretical mean values and standard deviation limits for $\sigma_b/\bar{b} = 0{\cdot}1$;

— deterministic solution.

The dispersion of the experimental transient responses confirms the necessity of including the fluctuations of the parameter b into the theoretical model in order to attain satisfactory agreement. The reason of this behaviour has to be looked for in the experimental method itself. It has been confirmed also by the experiments in "deterministic" arrangement, *i.e.* with constant parameter a.

In this case there results from Eqs (*22*), (*24*), (*26*) for all probability distributions considered

$$(\sigma_{yi}/\bar{b})^2 = (\sigma_b/\bar{b})^2 \left(1 - 2 \exp\left(-\bar{a}\, i\, \Delta t\right) + \exp\left(-2\bar{a}\, i\, \Delta t\right)\right). \qquad (28)$$

As can be seen from Fig. 6 the dispersion of experimental data fitts rather well Eq. (*28*), however the value $(\sigma_b/\bar{b}) = 0{\cdot}05$ has to be selected. This value is however considerably lower than would result from the results given in Figs 4, 5. The same effect can be observed also at the experiments for $\bar{a}\,\Delta t = 0{\cdot}2$.

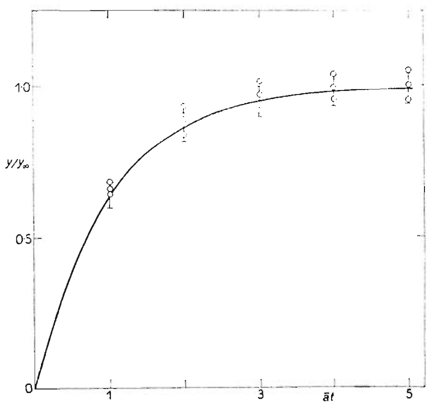

Fig. 6

Comparison of Experimental and Theoretical Values for $\sigma_a/\bar{a} = 0$

O-O-O Sample average and standard deviation values from experiments;

I theoretical standard deviation limits for $\sigma_b/\bar{b} = 0{\cdot}05$;

— deterministic step response.

The origin of this has to be looked for in the physical nature of the fluctuations of parameter b, in the corresponding Eq. (27) this being equal to the ratio K_2/K_1. Possible explanation can be seen in conductivity method *e.g.* in change of calibration constants K_2, K_1 from one experiment to another as caused *e.g.* by electrodes aging, electronic equipment drifts *etc*. The existing difference in the magnitude of the dispersion of parameter b however proves, that the mechanism of fluctuations is directly influenced also by the fluctuations in flow rate Q.

CONCLUSIONS

The extension of the analysis of the first order stochastic system for the case of two random parameters as applied to the transient response demonstrated that the dispersion of the transient response is influenced considerably more by the fluctuations in constant b, than by the fluctuations of the same relative magnitude of the time constant a. The distribution of the fluctuations of the time constant a has significant effect on the dispersion of transient response at higher values of $\bar{a}\ \Delta t$, it means at the fluctuations of flow rate with characteristic period comparable with the time constant of system. For the set of experiments which have been carried out so far there does not exist definite idea about the nature of statistical behaviour of constant b. Most probably it relates to the conductivity method used.

LIST OF SYMBOLS

a	parameter in Eq. (*1*)
b	parameter in Eq. (*1*)
$c_1(t)$, $c_2(t)$	input, output concentration functions
$E_1(t)$, $E_2(t)$	input, output conductivity measurements signals
K_2, K_1	calibration constants in Eq. (*27*)
$p(.)$	probability density function
$x(t)$	input time function
$y(t)$	output time function
$D\{\}$	dispersion operator
$E\{\}$	expected (mean) value operator
σ	standard deviation
$\delta()$	Dirac function
—	average value
t	index

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